How to Progress a Database III☆

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Abstract

In a seminal paper, Lin and Reiter introduced a model-theoretic definition for the progression of a basic action theory in the situation calculus, and proved that it implies the intended properties. They also showed that this definition comes with a strong negative result, namely that for certain cases first-order logic is not expressive enough to correctly characterize the progressed theory and second-order axioms are necessary. However, they also considered an alternative simpler definition according to which the progressed theory is always first-order definable. They conjectured that this alternative definition is incorrect in the sense that the progressed theory is too weak and may sometimes lose information. This conjecture and the status of the definability of progression in first-order logic has remained open since. In this paper we present two significant results about this alternative definition of progression. First, we prove the Lin and Reiter conjecture by presenting a case where the progressed theory indeed does lose information, thus closing a question that has remained open for more than ten years. Second, we prove that the alternative definition is nonetheless correct for reasoning about a large class of sentences, including some that quantify over situations.

☆This paper revises and combines results that first appeared in [1]. The title follows the line of work of Lin and Reiter [2, 3].

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1. Introduction

The situation calculus is a logical language that is specially designed for reasoning about action and change [4, 5]. A basic action theory is a situation calculus theory that describes what holds initially in the world as well as how the world evolves under the effects of actions. An example of a basic action theory is one that captures the dynamics of a board game: part of the theory, the initial knowledge base, describes the initial positions of the pieces on the board, and the rest of the theory characterizes the legal moves of the game and the effects (and non-effects) of performing those moves.

A fundamental problem in the field of reasoning about action and change is to determine whether or not some condition holds after a given sequence of actions has been performed. In other words, we start in an initial situation S_0, we perform a sequence of actions $\alpha_1, \ldots, \alpha_n$ taking us to a new situation S_n, and we wish to know if the condition holds in S_n. There are in fact two versions of this problem. The special case where the condition only refers to S_n is called the (simple) *projection problem* [5]. For example, we might want to know if a game piece is at a certain location after the moves $\alpha_1, \ldots, \alpha_n$ have been performed. The more general case is where the condition may refer to situations *in the future of* S_n. For example, we might want to know if a game piece can *ever* get to a certain location after these moves have been performed. This sort of reasoning, which we call the *generalized projection problem*, is a prerequisite to other forms of reasoning in dynamic domains such as planning and high-level program execution [6].

The simple projection problem can be solved by *regression* or by *progression* [7]. Roughly speaking, regression involves taking the condition about S_n and transforming it to an equivalent one about S_0 where we can use the initial knowledge base to answer the question; progression, on the other hand, involves replacing the initial knowledge base in the basic action theory by a new knowledge base that captures the facts that hold in S_n.

For the generalized problem, where the condition may refer to the future of S_n, the case is less clear. A model-theoretic definition of progression in the situation calculus that does the trick was first proposed by Lin and Reiter

[3]. However, their definition comes with a strong negative result: for certain kinds of basic action theories, first-order logic is not expressive enough and second-order logic is needed. Nonetheless, their result did not preclude the possibility of other forms of progression that could still allow us to solve the generalized problem while remaining first-order definable. In particular, one possible candidate for the new knowledge base is the infinite set of all those first-order sentences about S_n that are entailed by the original theory.

While this alternative first-order definition of progression clearly captures what holds in S_n, it is not clear whether it is sufficient to characterize the future of S_n, even in combination with the rest of the basic action theory. Lin and Reiter conjectured that this form of progression was too weak. It has been an open problem whether this conjecture is true or false, rendering unclear also the question whether there can be a correct progression that solves the generalized problem and is also first-order definable.

In this paper we present two major results. First, we give a proof for the Lin and Reiter conjecture: a progression based on the alternative first-order definition is indeed too weak for characterizing the future of S_n. We provide a basic action theory and a sentence about the future of S_n that demonstrate this. This is a major result that further supports the claim by Lin and Reiter that the progression of unrestricted basic action theories cannot be formalized correctly in first-order logic.

The second result is more positive. Lin and Reiter showed that the alternative first-order progression is correct for the simple projection problem [3]. Here we prove that it is also correct for a much wider class of sentences including sentences of the form "after α, property ϕ will always be true." This result establishes that this form of progression is actually more useful than what was originally believed.

The rest of the paper is organized as follows. In Sections 2 and 3 we review some preliminaries about the situation calculus and the basic action theories. In Section 4 we introduce the notion of a *correct progression* and define *strong progression*, an adapted formulation of the definition of progression by Lin and Reiter. Sections 5 and 6 contain the main contributions of this paper. In Section 5 we introduce *weak progression*, a formulation of the alternative first-order definition of progression, and give a proof for the conjecture by Lin and Reiter that weak progression is not correct in the general case (Theorem 2). In Section 6 we show that weak progression is correct when we restrict our attention to a wide class of sentences (Theorem 3). In Section 7 we review related work and in Section 8 we give some concluding remarks about the consequences of our results.

Finally, in Appendix A we discuss some subtle details about the definitions for progression that appear in [3] and [5], and we illustrate a problem in the definition of [5], while in Appendix B and Appendix C we give the proof of some lemmas that are long and tedious.

2. Situation calculus

The language $\mathcal{L}$ of the situation calculus as presented by Reiter [5] is a three-sorted first-order logic language with equality and some limited second-order features. The three sorts are the following: *action* for actions, *situation* for situations, and a catch-all sort *object* for everything else depending on the domain of application.

Similar to a normal one-sorted first-order language, $\mathcal{L}$ includes function and predicate symbols. In this case since there are three sorts, each of the symbols has a type that specifies the sorts for the arguments it takes. The situation calculus includes symbols only of certain types each of which has a special role in the representation of the world and its dynamics. One thing to note before moving to the formal details is that a situation is used to represent a world history as a sequence of actions, and the symbols that take arguments of sort *situation* are used to formalize the dynamics of the world.

The language of the situation calculus $\mathcal{L}$ includes the logical symbols $\neg, \wedge, \exists$, the symbol of equality $=$, and the following non-logical symbols:

- a countably infinite supply of variables for each of the three sorts, as well as a countably infinite supply of (second-order) predicate variables of all arities;
- a finite or countably infinite number of constant symbols of sort *object*;
- for each $n \geq 1$, a finite or countably infinite number of *object function* symbols, or simply *function* symbols, of type $(\mathit{action} \cup \mathit{object})^n \rightarrow \mathit{object}$;
- for each $n \geq 0$, a finite or countably infinite number of *action function* symbols of type $(\mathit{action} \cup \mathit{object})^n \rightarrow \mathit{action}$;
- the special *situation function* symbol $\mathit{do} : \mathit{action} \times \mathit{situation} \rightarrow \mathit{situation}$ and the constant S_0
- for each $n \geq 0$, a finite or countably infinite number of *predicate* symbols of type $(\mathit{action} \cup \mathit{object})^n$;

- the special predicate symbols $Poss$: $action \times situation$ and $\sqsubset$: $situation \times situation$;
- for each $n \geq 0$, a finite or countably infinite number of *relational fluent* symbols of type $(action \cup object)^n \times situation$.

The terms of the language are defined inductively similarly to a normal one-sorted language but also respecting the type of each symbol with respect to the three different sorts. We adopt the following notations with subscripts and superscripts: α and a for terms and variables of sort *action*; σ and s for terms and variables of sort *situation*; t and x, y, z, w for terms and variables of sort *object*. Also, we will use A for action functions, F, G for relational fluents, and b, c, d, e for constants of sort *object*.

An action term or simply an *action* represents an atomic action that may be performed in the world. For example consider the action $move(x, y)$ that may be used to represent that item x is moved to location y. A situation term or simply a *situation* represents a world history as a sequence of actions. The constant S_0 is used to denote the *initial situation* where no actions have occurred. Sequences of actions are built using the function symbol do, such that $do(\alpha, \sigma)$ represents the successor situation resulting from performing action α in situation σ.

A *relational fluent* is a predicate whose last argument is a situation, and thus whose truth value can change from situation to situation. For example, $At(x, y, \sigma)$ may be used to represent that item x is at location y in situation σ. In order to simplify the analysis we have restricted $\mathcal{L}$ so that there are no functional fluent symbols in $\mathcal{L}$, that is, functions whose last argument is a situation. This is not a restriction on the expressiveness of $\mathcal{L}$ as functional fluents can be represented by relational fluents with a few extra axioms. The normal predicates and functions that do not take arguments of sort *situation* are used to represent relations and functions that are *rigid* and remain the same for all situations.

Actions need not be executable in all situations, and the predicate atom $Poss(\alpha, \sigma)$ states that action α is executable in situation σ. For example, $Poss(move(x, y), \sigma)$ is intended to represent that the action $move(x, y)$ is possible in situation σ. Finally, the binary predicate symbol $\sqsubset$ provides an ordering on situations. The atom $\sigma \sqsubset \sigma'$ means that the action sequence σ' can be obtained from the sequence σ by performing one or more actions in σ. We will typically use the notation $\sigma \sqsubseteq \sigma'$ as a macro for $\sigma \sqsubset \sigma' \vee \sigma = \sigma'$.

The well-formed first-order formulas of $\mathcal{L}$ are defined inductively similarly to a normal one-sorted language but also respecting that each parameter has a unique sort. As far as the second-order formulas of $\mathcal{L}$ are concerned, only quantification over relations is allowed and the well-formed formulas are defined inductively similarly to a normal second-order language. The semantics of the situation calculus language is the standard model-theoretic Tarskian semantics. We assume that the reader is familiar with the notions of a *structure*, a *model*, *satisfaction* in a structure, and *entailment*. For the formal definitions the reader is referred to one of the standard textbooks for mathematical logic, such as [8] and [9]. Finally, to avoid confusion we note that whenever we say that two formulas are logically equivalent we assume that the logical symbol $=$ is always interpreted as the true identity.

Now we move on to see the specifics of the basic action theories, a special kind of situation calculus theories that represent the world and its dynamics.

3. Basic action theories

The language of the situation calculus provides the vocabulary that is needed to represent how properties of the world change under the effect of actions: the changing properties are represented as fluents, which are conditioned on a situation argument, and the dynamics is represented using rules that specify how the truth value of each fluent changes from any situation s to $do(a, s)$, i.e., the situation after the action a has been performed.

This representation task is tricky and requires solving a few problems that have been examined extensively in the literature, such as the *qualification problem*, the *ramification problem*, and the *frame problem* [4]. We will be dealing with situation calculus theories of a special kind, the so-called *basic action theories* [5], that provide an effective solution to the frame problem and a simple solution to the qualification problem that works for many practical scenarios. These theories consist mainly of two parts: i) a set of logical formulas that represent the initial state of the world and ii) a set of logical rules that represent how certain facts about the world change when actions are performed.

Before we proceed to the formal definition of a basic action theory we need to define the formulas that are *uniform in σ* as follows.

Definition 1 (Lin and Reiter 1997, Reiter 2001). For any situation term σ, we define $\mathcal{L}_\sigma$ to be the subset of the well-formed formulas of $\mathcal{L}$

(both first-order and second-order) that do not mention the predicates *Poss* or $\sqsubset$, do not quantify over variables of sort situation, do not mention equality on situations, and whenever they mention a term of sort situation in the situation argument position of a fluent, then that term is σ. When a formula $\phi(\sigma)$ is in $\mathcal{L}_\sigma$ we say that it is *uniform in* σ [5].

The definition of a basic action theory follows. Note that for the sake of readability we will typically omit the leading universal quantifiers in the axioms of the theory.

Definition 2 (Reiter 2001). A *basic action theory* $\mathcal{D}$ is a theory of the situation calculus language $\mathcal{L}$ of the form:[2]

$$\mathcal{D} = \mathcal{D}_{ap} \cup \mathcal{D}_{ss} \cup \mathcal{D}_{una} \cup \mathcal{D}_0 \cup \mathcal{D}_{\mathit{fnd}},$$

where each of the parts of $\mathcal{D}$ is as follows.

1. $\mathcal{D}_{ap}$ is a set of *action precondition axioms (APs)*, one for each action function symbol A, of the following form:

   $$\mathit{Poss}(A(\vec{x}), s) \equiv \Pi_A(\vec{x}, s),$$

   where $\Pi_A(\vec{x}, s)$ is a first-order formula uniform in s that does not mention any free variable other than $\vec{x}, s$. The action precondition axiom defines the preconditions to the executability of an action in a given situation in terms of properties holding in that situation alone.
2. $\mathcal{D}_{ss}$ is a set of *successor state axioms (SSAs)*, one for each relational fluent symbol F, of the following form:

   $$F(\vec{x}, do(a, s)) \equiv \Phi_F(\vec{x}, a, s),$$

   where $\Phi_F(\vec{x}, a, s)$ is a first-order formula uniform in s that does not mention any free variable other than $\vec{x}, a, s$. The successor state axiom for the fluent symbol F characterizes the conditions under which F has a specific truth value for $\vec{x}$ in the situation $do(a, s)$ as a function of the situation s.

[2] We slightly deviate from the standard notation in [5] as we use $\mathcal{D}_0$ instead of $\mathcal{D}_{S_0}$ and $\mathcal{D}_{\mathit{fnd}}$ instead of Σ. In the first case we use $\mathcal{D}_0$ in order to avoid using a symbol with a subscript (S_0) as a subscript of $\mathcal{D}$. In the second case we choose to use $\mathcal{D}_{\mathit{fnd}}$ for purposes of uniformity in the parts of $\mathcal{D}$. Finally, note that even though we do not use a different symbol like Reiter [5], here it is also the case that $\mathcal{D}_{\mathit{fnd}}$ is identical in all basic action theories.

3. $\mathcal{D}_{una}$ is the set of unique-names axioms for actions: $A(\vec{x}) \neq A'(\vec{y})$, and $A(\vec{x}) = A(\vec{y}) \supset \vec{x} = \vec{y}$, for each pair of distinct action function symbols A and A'.
4. $\mathcal{D}_0$ is a set of first-order sentences uniform in S_0 that describe the state of the world in the initial situation when no action has been performed. We will typically refer to this set as the *initial knowledge base (KB)*.
5. $\mathcal{D}_{fnd}$ is the following set of domain independent foundational axioms that formally define the space of situations and the ordering $\sqsubset$:

$$do(a,s) = do(a',s') \supset (a = a' \wedge s = s')$$

$$\forall P\big(P(S_0) \wedge \forall a \forall s\big(P(s) \supset P(do(a,s))\big) \supset \forall s P(s)\big)$$

$$\neg s \sqsubset S_0$$

$$s \sqsubseteq do(a,s') \equiv s \sqsubset s' \vee s = s'.$$

Note that $\mathcal{D}_{fnd}$ is the only place where a second-order axiom is used, namely the second axiom that quantifies over the second-order predicate variable P. The purpose of this axiom is to ensure that the domain of situations is the smallest set that includes the initial situation and situations that are built using the function *do*, thus ensuring that when we quantify over situations we only refer to situations that are reachable from S_0 by a finite number of applications of the function *do*.

We now proceed to define the problem of progression for such theories.

4. The problem of progression

The progression of a basic action theory is the problem of *updating* the initial knowledge base so that it reflects the current state of the world after some actions have been performed, instead of the initial state of the world. In other words, in order to do a one-step progression of the basic action theory $\mathcal{D}$ with respect to the ground action α we need to replace $\mathcal{D}_0$ in $\mathcal{D}$ by a suitable set $\mathcal{D}_\alpha$ of sentences so that the original theory $\mathcal{D}$ and the theory $(\mathcal{D} - \mathcal{D}_0) \cup \mathcal{D}_\alpha$ are equivalent with respect to how they describe the situation $do(\alpha, S_0)$ and the situations in the future of $do(\alpha, S_0)$ [3]. For readability reasons we will typically use S_α to denote the situation term $do(\alpha, S_0)$.

As we will be interested in identifying classes of sentences that can be correctly evaluated by the updated theory $(\mathcal{D} - \mathcal{D}_0) \cup \mathcal{D}_\alpha$, we introduce the notion of a *correct progression* as follows.

Definition 3 (Correct progression). Let $\mathcal{D}$ be a basic action theory, α a ground action term, and $\mathcal{D}_\alpha$[3] a set of (first-order or second-order) sentences uniform in S_α. We say that $\mathcal{D}_\alpha$ is a *correct progression* of $\mathcal{D}_0$ wrt α, $\mathcal{D}$, and the set of sentences $\mathcal{Z}$ iff for every sentence ϕ in $\mathcal{Z}$,

$$\mathcal{D} \models \phi \text{ iff } (\mathcal{D} - \mathcal{D}_0) \cup \mathcal{D}_\alpha \models \phi.$$

When we progress $\mathcal{D}_0$ wrt to an action α, we want to find a progression $\mathcal{D}_\alpha$ that is correct wrt a set $\mathcal{Z}$ that includes all those first-order sentences whose truth value depends only on S_α and situations in the future of S_α.

In a seminal paper Lin and Reiter [3] gave a model-theoretic definition for the progression $\mathcal{D}_\alpha$ of $\mathcal{D}_0$ wrt α and $\mathcal{D}$ that achieves this goal. Their definition essentially requires for the two theories $\mathcal{D}$ and $(\mathcal{D}-\mathcal{D}_0)\cup\mathcal{D}_\alpha$ that any model of one is indistinguishable from some model of the other with respect to how they interpret the situations in S_α and the future of S_α. Using this property then one can prove that a set $\mathcal{D}_\alpha$ that follows their definition of progression is correct wrt any set of first-order sentences whose truth value depends only on S_α and situations in the future of S_α.

In the same paper Lin and Reiter showed that if we assume a finite number of predicates, functions, and fluents in $\mathcal{L}$, and a finite $\mathcal{D}_0$, then we can always use second-order logic to specify a set $\mathcal{D}_\alpha$ that qualifies as a progression according to their definition. In particular, Lin and Reiter showed how to construct a second-order sentence which along with the set $\mathcal{D}_{una}$ is a progression according to their model-theoretic definition.

As it is often easier to work with a second-order sentence than a model-theoretic relation between logical theories, we chose to use a definition of progression that is based on this result by Lin and Reiter. Our definition, namely *strong progression*, is essentially the original definition of progression by Lin and Reiter [3] slightly adapted to comply with the newer form of basic action theories that do not use *Poss* in the successor state axioms, and expressed in the second-order counterpart that was identified by Lin and Reiter. For a more detailed discussion about the syntactic and aesthetic differences between the definitions of progression as they appear in [3], [5], and in this paper, the reader is referred to Appendix A.

[3] Similar to what we did in Definition 2 we slightly deviate from the notation of Lin and Reiter [3] and use $\mathcal{D}_\alpha$ instead of $\mathcal{D}_{S_\alpha}$ in order to avoid using a symbol with a subscript as a subscript of $\mathcal{D}$.

We now go over the intuition behind the second-order characterization of progression, and introduce the notation that we will use in our definition.

Let $F_1, \ldots, F_n$ be the fluent symbols of $\mathcal{L}$, $\mathcal{D}$ be a basic action theory with a finite $\mathcal{D}_0$, and α a ground action term. Also assume that the successor state axiom for F_i is of the form $F_i(\vec{x}, do(a, s)) \equiv \Phi_i(\vec{x}, a, s)$. We want to find a set $\mathcal{D}_\alpha$ that successfully describes the situation S_α. Observe that $\mathcal{D}$ already tells us what is known about the situation S_α: $\mathcal{D}_0$ tells what is known about S_0, and the successor state axioms tell us how each fluent changes in going from S_0 to S_α. So in a sense, the set

$$\mathcal{D}_0 \cup \Big\{ \bigwedge_{i=1}^{n} \forall \vec{x}. F_i(\vec{x}, S_\alpha) \equiv \Phi_i(\vec{x}, \alpha, S_0) \Big\}$$

qualifies as the set $\mathcal{D}_\alpha$ we are looking for, except for the fact that it also includes what is known about S_0, therefore is not uniform in S_α. The progression we propose removes the dependency on S_0 by using second-order quantification over predicates in order to express the information about S_0, instead of using the original set $\mathcal{D}_0$ as is. The resulting sentence is then uniform in S_α. More precisely, we introduce the following notation that is similar to the so-called second-order *lifting* of Lin and Reiter [3].

Definition 4. Let $F_1, \ldots, F_n$ be relational fluent symbols, and $Q_1, \ldots, Q_n$ be second-order (non-fluent) predicate variables. For any first-order formula ϕ in $\mathcal{L}$, let $\phi\langle \vec{F} : \vec{Q} \rangle$ be the formula that results from replacing any fluent atom $F_i(t_1, \ldots, t_n, \sigma)$ in ϕ, where σ is a situation term, by $Q_i(t_1, \ldots, t_n)$.

The trick here is that we drop situation terms from all fluent atoms that are mentioned in ϕ and replace each fluent atom with a second order predicate variable. The resulting formula is then uniform in any situation term. We will use this notation as follows to define a second-order sentence uniform in S_α that will be the basis of our definition for progression.

Definition 5. Let $F_1, \ldots, F_n$ be all the relational fluent symbols of $\mathcal{L}$ and $\mathcal{D}$ a basic action theory, where $\mathcal{D}_0$ is a finite set of first-order sentences, ϕ is the conjunction of the sentences in $\mathcal{D}_0$, and for all i, $1 \leq i \leq n$, the successor state axiom for F_i has the form $F_i(\vec{x}, do(a, s)) \equiv \Phi_i(\vec{x}, a, s)$. Let α be a ground action term, and $Q_1, \ldots, Q_n$ be predicate variables. Then, $Pro(\mathcal{D}, \alpha)$ is the following second-order sentence uniform in S_α:

$$\exists \vec{Q}.\ \phi\langle \vec{F} : \vec{Q} \rangle \ \wedge\ \bigwedge_{i=1}^{n} \forall \vec{x}. F_i(\vec{x}, S_\alpha) \equiv (\Phi_i(\vec{x}, \alpha, S_0)\langle \vec{F} : \vec{Q} \rangle).$$

Now we are ready to give the precise definition of *strong progression*.

Definition 6 (Strong progression). Let $\mathcal{L}$ be a situation calculus language with a finite number of predicate, function, and fluent symbols, $\mathcal{D}$ a basic action theory over $\mathcal{L}$ with a finite $\mathcal{D}_0$, α a ground action term, and $\mathcal{D}_\alpha$ a set of sentences uniform in S_α. The set $\mathcal{D}_\alpha$ is a *strong progression* of $\mathcal{D}_0$ wrt α and $\mathcal{D}$ iff $\mathcal{D}_\alpha \cup \mathcal{D}_{una}$ is logically equivalent[4] to $\{Pro(\mathcal{D}, \alpha)\} \cup \mathcal{D}_{una}$.

Note that $\mathcal{D}_\alpha$ is unique up to logical equivalence *assuming that actions have unique-names*. The reason why we don't use the simpler condition that $\mathcal{D}_\alpha$ be logically equivalent to $Pro(\mathcal{D}, \alpha)$, is that $\mathcal{D}_{una}$ is needed in order to get a correct characterization of the situation S_α but since it is already included in $\mathcal{D}$ we don't want to assume that $\mathcal{D}_{una}$ is also included in $\mathcal{D}_\alpha$.

As far as the correctness of a strong progression $\mathcal{D}_\alpha$ is concerned, following the proof of result [3, Theorem 2] we can show for the two theories $\mathcal{D}$ and $(\mathcal{D} - \mathcal{D}_0) \cup \mathcal{D}_\alpha$ that any model of one is indistinguishable from some model of the other with respect to how they interpret the situations in S_α and the future of S_α, and use this property to prove that $\mathcal{D}_\alpha$ is correct wrt any set of first-order sentences whose truth value depends only on S_α and situations in the future of S_α. In particular, the following theorem is a reformulation of result [3, Theorem 1] expressed in terms of our definition of strong progression.

Theorem 1 (Lin and Reiter 1997). *Let $\mathcal{L}$ be a situation calculus language with a finite number of predicate, function, and fluent symbols, and $\mathcal{D}$ a basic action theory over $\mathcal{L}$ such that $\mathcal{D}_0$ is finite. Let α a ground action term, and $\mathcal{D}_\alpha$ a strong progression of $\mathcal{D}_0$ wrt α and $\mathcal{D}$. Then, for every sentence ϕ uniform in S_α (first-order or second-order), $\mathcal{D} \models \phi$ iff $\mathcal{D}_\alpha \cup \mathcal{D}_{una} \models \phi$.*

This result implies the following for the correctness of strong progression according to Definition 3.

Corollary 1 (Strong progression is correct wrt $\mathcal{L}_{S_\alpha}$). *Let $\mathcal{L}$, $\mathcal{D}$, and α as in Theorem 1, and $\mathcal{D}_\alpha$ a strong progression of $\mathcal{D}_0$ wrt α and $\mathcal{D}$. Then, $\mathcal{D}_\alpha$ is a correct progression of $\mathcal{D}_0$ wrt α, $\mathcal{D}$, and $\mathcal{L}_{S_\alpha}$, where $\mathcal{L}_{S_\alpha}$ is the set of (first-order or second-order) sentences uniform in $do(\alpha, S_0)$.*

[4] As we mentioned earlier, whenever we say that two formulas are logically equivalent we assume that the logical symbol $=$ is always interpreted as the true identity.

Although the definition of strong progression is formulated in second-order logic, like Lin and Reiter we are concerned with finding progressions that can be expressed in first-order logic. It is not hard to see that in simple cases that are used in practice (such as STRIPS planning [10]), this definition does the right thing and remains within first-order logic. However, as it was first shown in [3], there are cases of basic action theories where no first-order strong progression exists. In the next section we examine the first-order definability of a progression that is also correct in the general case.

We close this section with a remark about the requirement that $\mathcal{D}_\alpha$ should be uniform in S_α. Strictly speaking the new theory $(\mathcal{D} - \mathcal{D}_0) \cup \mathcal{D}_\alpha$ is not a basic action theory according to Definition 2 because the updated knowledge base $\mathcal{D}_\alpha$ is not uniform in S_0. Nonetheless, getting a basic action theory in the formal sense is a simple matter of replacing S_α by S_0 in all the sentences in $\mathcal{D}_\alpha$. The reason why $\mathcal{D}_\alpha$ is typically assumed in the literature to be uniform in S_α is that it simplifies the analysis, as we don't need to change our "S_0 point of reference" when we examine the original and the progressed theory.

5. On the first-order definability of progression

Even though the definition of strong progression uses second-order logic, we are interested in finding a first-order set $\mathcal{D}_\alpha$ that qualifies as a strong progression of $\mathcal{D}_0$. In some cases this is feasible but as it was first shown by Lin and Reiter [3] there are cases where no first-order strong progression exists. It has been unclear whether this is a problem of the particular *definition* of strong progression or if it is an inherent difficulty of the problem of progression. In other words, it has been an open question whether there is an alternative (weaker) definition for $\mathcal{D}_\alpha$ according to which $\mathcal{D}_\alpha$ is always first-order definable and is also correct in the general case.

In fact there is a straightforward alternative definition for $\mathcal{D}_\alpha$ that is always first-order. The idea is to let $\mathcal{D}_\alpha$ be the infinite set of first-order sentences uniform in S_α that are entailed by $\mathcal{D}$ [11]. The intuition is that if we replace $\mathcal{D}_0$ by a set $\mathcal{D}_\alpha$ that is strong enough to entail all the first-order sentences uniform in S_α that the original theory entails, then it should follow that $(\mathcal{D} - \mathcal{D}_0) \cup \mathcal{D}_\alpha$ also entails the same first-order sentences about the future of S_α as the original theory $\mathcal{D}$. We call this alternative definition of progression *weak progression* and in order to avoid confusion with strong progression we will be using $\mathcal{F}_\alpha$ to refer to a weak progression of $\mathcal{D}_0$ wrt α.

Definition 7 (Weak progression). Let $\mathcal{D}$ be a basic action theory, α a ground action term, and $\mathcal{F}_\alpha$ a set of first-order sentences uniform in S_α. The set $\mathcal{F}_\alpha$ is a *weak progression* of $\mathcal{D}_0$ wrt α and $\mathcal{D}$ iff for all first-order sentences ϕ uniform in S_α, $\mathcal{F}_\alpha \cup \mathcal{D}_{una} \models \phi$ iff $\mathcal{D} \models \phi$.

If we could prove that a weak progression of $\mathcal{D}_0$ is correct in the general case then the this definition would be the preferred option, as strong progression is much more cumbersome to work with and also comes with the strong negative result that second-order logic may be necessary in some cases. Following intuitions and results in [12] Lin and Reiter conjectured that we can find counter examples which show that a weak progression is not correct in the general case. In this section we will identify such an example and prove the conjecture, thus showing that weak progression is indeed too weak.

In order to be precise in our formulation of the conjecture and our proof, we will now introduce the set $\mathcal{L}_\sigma^F$ that includes a wide range of first-order sentences whose truth value depends only on σ and situations in the future of σ. For a sentence ϕ in $\mathcal{L}_\sigma^F$ we will typically say that ϕ is *about the future of* σ. Note though that we do not imply that any first-order property which is about the future σ can be represented as a sentence in $\mathcal{L}_\sigma^F$. This is essentially a "wide enough" set that we will be using in this section to prove the conjecture by showing that in general a weak progression is not correct wrt $\mathcal{L}_\sigma^F$.

We first introduce some notation to specify sequences of actions and situation terms that are *rooted* at some other situation term, similarly to [13, 5].

Definition 8 (Reiter 2001, Gabaldon 2002). Let σ be a situation term and δ be a (possibly empty) vector of action terms $\langle \alpha_1, \ldots, \alpha_n \rangle$. We use $do(\delta, \sigma)$ to denote the following situation: $do(\alpha_n, do(\alpha_{n-1}, \ldots do(\alpha_1, \sigma) \ldots))$. We say that a situation term κ is *rooted at* σ iff κ is syntactically the same term as $do(\delta, \sigma)$, for some vector of action terms δ.

The intuition is that a situation term κ is rooted at some other situation term σ iff κ can be obtained from σ by "adding" a sequence of actions using the function *do*.

We will also need to restrict our attention to formulas that only refer to some situation term σ and possible futures of σ. For this purpose we introduce the next definition using the notion of rooted situation terms.

Definition 9 ($\mathcal{L}_\sigma^F$). Let σ, κ be situation terms and ϕ a rectified[5] formula in $\mathcal{L}$. We say that κ *is in the future of* σ *in* ϕ iff one of the following holds:[6]

- κ is σ, or
- κ is rooted at some situation term κ' that is in the future of σ in ϕ, or
- κ is a variable and $\forall\kappa(\kappa' \sqsubseteq \kappa \supset \beta)$ or $\exists\kappa(\kappa' \sqsubseteq \kappa \wedge \beta)$ is a sub-formula of ϕ, where κ' is a situation term that is in the future of σ in ϕ.

We say that the formula ϕ is *about the future of* σ iff the situation terms in ϕ that appear as arguments of *Poss* or some fluent or the equality predicate are all in the future of σ in ϕ. We define $\mathcal{L}_\sigma^F$ as the set of all rectified sentences ϕ in $\mathcal{L}$ such that ϕ is about the future of σ.

The intuition is that if a sentence is about the future of S_α then its truth depends only on S_α and situations that come after S_α. An example follows.

Example 1. Let $\phi(s)$ be a (first-order or second-order) formula uniform in s. Then the following sentence is about the future of S_α and expresses that $\phi(s)$ holds in all situations that are rooted at S_α: $\forall s(S_\alpha \sqsubseteq s \supset \phi(s))$. The third point in the previous definition specifies a way that sub-formulas about the future may be combined in an inductive way. For instance, the following sentence is about the future of S_α as well and expresses that for every situation s rooted at S_α there is a situation s' rooted at s such that $\phi(s')$ holds: $\forall s(S_\alpha \sqsubseteq s \supset \exists s'(s \sqsubseteq s' \wedge \phi(s')))$. Finally, note that none of the sentences we examined in this example are uniform in any situation term σ.

We can now state the conjecture by Lin and Reiter [3] in an equivalent way using the terminology that we have introduced in this paper.

Conjecture 1 (Lin and Reiter 1997). *There is a situation calculus language $\mathcal{L}$ with a finite number of predicate, function, and fluent symbols, a basic action theory $\mathcal{D}$ over $\mathcal{L}$ with a finite $\mathcal{D}_0$, and a ground action term α, such that the following holds: if $\mathcal{F}_\alpha$ is a weak progression of $\mathcal{D}_0$ wrt α and $\mathcal{D}$ then $\mathcal{F}_\alpha$ is not a correct progression wrt α, $\mathcal{D}$, and $\mathcal{L}_\sigma^F$, where σ is $do(\alpha, S_0)$.*

We now proceed to the proof of this conjecture.

[5] A formula is rectified iff no variable occurs both bound and free, and all quantifiers in the formula refer to different variables.

[6] Strictly speaking we only care about future situations s that are *executable* in the sense of satisfying this formula: $\forall a.\forall s^*.do(a, s^*) \sqsubseteq s \supset Poss(a, s^*)$. A more refined definition would include this constraint but is not necessary for our analysis.

5.1. Weak progression is not correct wrt $\mathcal{L}^F_{S_\alpha}$

In this section we give a proof of Conjecture 1 thus resolving the open question whether a weak progression is a correct progression in the general case. In particular, we will present a basic action theory $\mathcal{D}$ and a weak progression $\mathcal{F}_\alpha$ such that $(\mathcal{D} - \mathcal{D}_0) \cup \mathcal{F}_\alpha$ fails to entail a sentence ϕ^* about the future of S_α that is nonetheless entailed by the original theory $\mathcal{D}$. The proof is based on the notion of *unnamed objects* that we will be defining shortly. We start by presenting the basic action theory that we will use for the proof.

Definition 10 (The infinite doors domain). Let $\mathcal{L}$ be the situation calculus language that consists of the standard logical symbols and the symbols $Poss, do, S_0$, the fluent $F(x, s)$, the action constants A, B, the object function $n(x)$, and the object constant 0. Let $\mathcal{D} = \mathcal{D}_{ap} \cup \mathcal{D}_{ss} \cup \mathcal{D}_{una} \cup \mathcal{D}_0 \cup \mathcal{D}_{fnd}$ be the basic action theory of the *infinite doors domain*, where each of the parts of $\mathcal{D}$ is as follows.

1. $\mathcal{D}_{ap}$ consists of following two sentences:

$$Poss(A, s) \equiv true, \tag{1}$$

$$Poss(B, s) \equiv true. \tag{2}$$

2. $\mathcal{D}_{ss}$ consists of the following sentence:

$$\begin{aligned} F(x, do(a, s)) \equiv\ & a = A \wedge x = 0\ \vee \\ & a = B \wedge \neg F(x, s) \wedge \exists y (x = n(y) \wedge F(y, s)). \end{aligned} \tag{3}$$

3. $\mathcal{D}_{una}$ consists of the following sentence:

$$A \neq B. \tag{4}$$

4. $\mathcal{D}_0$ consists of the following sentences:

$$\forall a (a = A \vee a = B) \tag{5}$$

$$\forall x (x \neq 0 \equiv \exists y\ n(y) = x) \tag{6}$$

$$\forall x \forall y (n(x) = n(y) \supset x = y) \tag{7}$$

$$F(0, S_0) \wedge \forall x\, (F(x, S_0) \supset F(n(x), S_0)) \tag{8}$$

$$\exists x\ \neg F(x, S_0) \tag{9}$$

5. $\mathcal{D}_{fnd}$ is as in Definition 2.

In the infinite doors domain there is an infinite number of doors that are located one next to the other. The fluent $F(x, s)$ represents that the object x, i.e., the door x, is open in the situation s. The door 0 is the first door in the chain and the function $n(x)$ denotes the door that is next to the door x. The sentences (6) and (7) in $\mathcal{D}_0$ ensure that for every door x, there is a unique door that is next to x and a unique door y such that x is next to y, except for door 0 that is the first door in the chain. Note then that all models of D_0 must have an infinite object domain and contain at least an infinite chain of doors.

The basic action theory $\mathcal{D}$ of the infinite doors domain was carefully defined so that all the models of $\mathcal{D}$ satisfy two properties that we will take advantage of in the sequel. Before we state the properties we need to introduce the notion of *named* and *unnamed* objects as follows.

Definition 11. Let $\mathcal{L}$ be the language of the infinite doors domain, GT the set of all the ground terms of sort object in $\mathcal{L}$, and M an $\mathcal{L}$-structure. For every q in the object domain of M we say that q is *named* iff there is a term t in GT such that t is interpreted as q in M, and *unnamed* otherwise. Also, we say that M is a *term structure* iff all the elements of the object domain of M are named.

The first property of $\mathcal{D}$ is due to the initial knowledge base $\mathcal{D}_0$ that can only be satisfied in models with at least one unnamed object. In particular, $\mathcal{D}_0$ is defined so that there exists a door that is different than any door in the infinite chain of doors that start from the door zero.

Lemma 1. *Let $\mathcal{D}$ be the basic action theory of the infinite doors domain. No model of $\mathcal{D}$ is a term structure.*

Proof. Observe that each of the ground terms of sort object in the language has the form $n^k(0)$, i.e., it is constructed by a finite number of applications of the function n to the constant 0. By induction on the construction of ground terms of sort object it follows that if M satisfies the sentence (8) then for all named objects q and an arbitrary variable assignment μ,

$$M, \mu^x_q \models F(x, S_0).$$

The sentence (9) on the other hand is satisfied only in a structure M that has an element q' in the object domain such that

$$M, \mu^x_{q'} \not\models F(x, S_0).$$

Therefore, the set $\{(8),(9)\}$ can only be satisfied in a structure that has an unnamed object. $\square$

The second property of $\mathcal{D}$ is related to the effects of the actions A and B. The action A affects all the doors in the domain changing their status so that in the resulting situation exactly one of them is open, namely the door 0. Every time a series of actions B is performed after A, in the resulting situation there is exactly one door that is open which is identified as follows.

Lemma 2. *Let $\mathcal{D}$ be the basic action theory of the infinite doors domain, M a model of $\mathcal{D}$, μ an arbitrary variable assignment, and S_A a macro for the situation term $do(A, S_0)$. For every action sequence δ, $M, \mu^x_q \models F(x, do(\delta, S_A))$ iff q is the denotation of $n^k(0)$, where k is the number of B actions that appear after the last occurrence of the action A in $\langle A, \delta\rangle$, and $n^0(0) = 0$.*

Proof. First, observe that the sentences (6) and (7) in $\mathcal{D}_0$ ensure the uniqueness of names for the ground terms of sort object. We will prove the lemma by induction on the number k of B actions that appear after the last occurrence of the action A in $\langle A, \delta\rangle$. The base case is when $k = 0$, which means that A is the last action in $\langle A, \delta\rangle$. In this case it follows by the successor state axiom for F, i.e., sentence (3), that F is false in $do(\delta, S_A)$ for all the elements of the object domain except for the denotation of 0, therefore the lemma holds. For the induction step we assume that the lemma holds for k and prove for $k + 1$. By the induction hypothesis it follows that

$$M, \mu^x_q \models F(x, do(\delta, S_A))$$

iff q is the denotation of $n^k(0)$. It follows that

$$M, \mu^{xy}_{qr} \models \neg F(x, do(\delta, S_A)) \wedge \exists y(x = n(y) \wedge F(y, do(\delta, S_A)))$$

iff r is the denotation of $n^k(0)$ and q is the denotation of $n^{k+1}(0)$. Therefore, by the successor state axiom for F it follows that after one more B action, F is false in the resulting situation $do(B, do(\delta, S_A))$ for all the elements of the object domain except for the denotation of $n^{k+1}(0)$. Therefore the induction holds. $\square$

We now present the sentence ϕ^* that we will use to prove Conjecture 1.

Definition 12. Let $\mathcal{L}$ be the language of the infinite doors domain, and S_A a macro for the situation term $do(A, S_0)$. ϕ^* is the following first-order sentence of $\mathcal{L}$:

$$\exists x \forall s (S_A \sqsubseteq s \supset \neg F(x, s)).$$

Sentence ϕ^* expresses that there is a door x such that after action A is performed x will remain closed forever. First, we show that the basic action theory $\mathcal{D}$ of the infinite doors domain entails ϕ^*. The intuition is that x is the unnamed object that necessarily exists in all models of $\mathcal{D}$.

Lemma 3. *Let $\mathcal{D}$ be the basic action theory of the infinite doors domain and ϕ^* the first-order sentence as in Definition 12. Then, $\mathcal{D} \models \phi^*$.*

Proof. Let M be a model of $\mathcal{D}$. By Lemma 2 it follows that for every situation in the future of S_A there can only be named objects for which F is true. By Lemma 1 it follows that there exists at least one unnamed object q in the domain. Therefore there is an x, namely the unnamed object q, such that $F(x, s)$ is false in every situation in the future of S_A, which implies that $M \models \phi^*$. Since M was arbitrary the lemma follows. □

Note that the sentence ϕ^* is in $\mathcal{L}^F_{S_A}$, and that the original theory $\mathcal{D}$ entails ϕ^*. We now proceed to show that a weak progression $\mathcal{F}_\alpha$ of $\mathcal{D}_0$ wrt A and $\mathcal{D}$ actually fails to entail ϕ^*. First, we identify a weak progression $\mathcal{F}_\alpha$.

Lemma 4. *Let $\mathcal{D}$ be the basic action theory of the infinite doors domain of Definition 10 and $\mathcal{F}_\alpha$ the following set of first-order sentences:*

$$\{\forall x (x = 0 \equiv F(x, S_A)),\ (5),\ (6),\ (7)\}.$$

Then, $\mathcal{F}_\alpha$ is a weak progression of $\mathcal{D}_0$ wrt the ground action A and $\mathcal{D}$.

The proof is long and tedious and can be found in the appendix. It involves a series of model-theoretic constructions using properties of first-order logic such as the upward Löwenheim-Skolem Theorem of first-order logic.

The set $\mathcal{F}_\alpha$ is an updated version of the initial knowledge base of the basic action theory $\mathcal{D}$ of the infinite doors domain. The only difference is that the sentences about F are replaced by a sentence that expresses that there is exactly one door that is open, namely the door zero. Intuitively this is what summarizes the effects of the action A as far as first-order entailment is concerned, and we would expect that this is sufficient for $\mathcal{F}_\alpha$ to be a correct progression. Indeed, there are many non-trivial sentences about the future of S_A that are entailed by $\mathcal{D}$ and $\mathcal{F}_\alpha$. For example, the sentence $\exists x \neg F(x, S_A)$, and for $j > 0$ the sentences $\exists x(\bigwedge_1^j x \neq n^k(0) \wedge \neg F(x, S_A))$, are all uniform in S_A and are entailed both by $\mathcal{D}$ and $\mathcal{F}_\alpha$.

Nonetheless, there is a property of $\mathcal{D}_0$ that persists in S_A which $\mathcal{F}_\alpha$ fails to express, namely that there is an unnamed object in every model. Of course, this is not reflected in any first-order sentence uniform in S_A that is entailed by $\mathcal{D}$ and this is why $\mathcal{F}_\alpha$, which is defined based on the first-order entailments of $\mathcal{D}$, fails to express it. Apparently, this property is reflected though in a first-order sentence that is about the future of S_A, namely the sentence ϕ^* of Definition 12. The following lemma shows that $\mathcal{F}_\alpha$ indeed fails to express this property and as a result $(\mathcal{D} - \mathcal{D}_0) \cup \mathcal{F}_\alpha$ fails to entail ϕ^*.

Lemma 5. *Let $\mathcal{D}$ be the basic action theory of the infinite doors domain, ϕ^* the first-order sentence as in Definition 12, and $\mathcal{F}_\alpha$ the set of first-order sentences as in Lemma 4. Then, $(\mathcal{D} - \mathcal{D}_0) \cup \mathcal{F}_\alpha \not\models \phi^*$.*

Proof. Consider a model M of $\mathcal{F}_\alpha$ that has the natural numbers as the domain for objects, and interprets the constant symbol 0 as the number zero and the object function symbol n as the successor function. Note that it is easy to verify that such a model exists by observing the four sentences that $\mathcal{F}_\alpha$ consists of. M is a term model where the ground term $n^k(0)$ is interpreted as the number $k \in \mathbb{N}$. Observe that the property about F that we showed in Lemma 2 also holds for all the models of $(\mathcal{D} - \mathcal{D}_0) \cup \mathcal{F}_\alpha$. It follows that for every x in the object domain there is a sequence of actions after which $F(x, s)$ becomes true: x is the denotation of some term $n^k(0)$, therefore $F(x, do(\delta, S_A))$ is true when δ is a sequence of B actions of size k. It follows that $M \models \forall x \exists s(S_A \sqsubseteq s \wedge F(x, s))$ which is equivalent to $M \models \neg\phi^*$. □

Before moving on to state the main result of this section, it is important to observe that a strong progression $\mathcal{D}_A$ actually captures the property that there is an unnamed object in every model of $\mathcal{D}$, and as a result $(\mathcal{D} - \mathcal{D}_0) \cup \mathcal{D}_\alpha$ entails ϕ^*. To see this more clearly, let us construct the sentence $Pro(\mathcal{D}, A)$ that is used in Definition 6. Recall that $Pro(\mathcal{D}, A)$ uses a set of second-order variables to represent the sentences in $\mathcal{D}_0$, and an instantiated version of the successor state axioms in $\mathcal{D}_{ss}$ in order to model the transition from S_0 to S_A. $Pro(\mathcal{D}, A)$ is in fact the following second-order sentence:

$$\exists Q.\, Q(0) \wedge \forall x(Q(x) \supset Q(n(x))) \wedge \exists x \neg Q(x) \wedge \psi \wedge \forall x(F(x, S_A) \equiv x = 0), \quad (10)$$

where ψ is the conjunction of the rigid sentences of $\mathcal{D}_0$, i.e., (5), (6), (7), and the successor state axiom is simplified to include only the disjunct that refers to action A. As it should be obvious now, this sentence cannot be simplified to $\psi \wedge \forall x(F(x, S_A) \equiv x = 0)$ which is the weak progression of Lemma 4.

The next theorem establishes that Conjecture 1 is indeed true and thus closes the open question about the correctness of weak progression.

Theorem 2 (Weak progression is not correct wrt $\mathcal{L}^F_{S_\alpha}$). *There is a basic action theory $\mathcal{D}$, a ground action term α, and a first-order sentence ϕ about the future of S_α such that $\mathcal{D} \models \phi$ but $(\mathcal{D} - \mathcal{D}_0) \cup \mathcal{F}_\alpha \not\models \phi$, where $\mathcal{F}_\alpha$ is a weak progression of $\mathcal{D}_0$ wrt α and $\mathcal{D}$.*

Proof. Let $\mathcal{D}$ be the basic action theory of the infinite doors domain, α be the constant action A, ϕ be the first-order sentence ϕ^* of Definition 12, and $\mathcal{F}_\alpha$ be the set of first-order sentences of Lemma 4. The theorem follows from Lemma 3, Lemma 4, and Lemma 5. □

This result shows a case where a weak progression $\mathcal{F}_\alpha$ of $\mathcal{D}_0$ wrt α and $\mathcal{D}$ is not a correct progression wrt $\mathcal{D}_0$ wrt α, $\mathcal{D}$, and $\mathcal{L}^F_\sigma$. Therefore it follows that weak progression is not correct in the general case. Moreover, it shows that any definition of progression that relies on the first-order sentences uniform in S_α that are entailed by $\mathcal{D}$ would also fail to capture the second-order property $\exists Q.\ Q(0) \wedge \forall x(Q(x) \supset Q(n(x))) \wedge \exists x \neg Q(x)$ that we identified in the infinite doors domain and in sentence (10).

We conclude with a lemma which shows that whenever a first-order strong progression exists, it is actually logically equivalent to a weak progression.

Lemma 6. *Let $\mathcal{D}$ be a basic action theory with a finite $\mathcal{D}_0$, α a ground action term, $\mathcal{D}_\alpha$ a strong progression of $\mathcal{D}_0$ wrt α and $\mathcal{D}$, and $\mathcal{F}_\alpha$ a weak progression of $\mathcal{D}_0$ wrt α and $\mathcal{D}$. Then $\mathcal{D}_\alpha \cup \mathcal{D}_{una}$ is logically equivalent to $\mathcal{F}_\alpha \cup \mathcal{D}_{una}$.*

Proof. By Definition 7 it follows that $\mathcal{F}_\alpha \cup \mathcal{D}_{una}$ is a set of first-order sentences uniform in S_α and $\mathcal{D} \models \mathcal{F}_\alpha \cup \mathcal{D}_{una}$. By Theorem 1 it then follows that

$$\mathcal{D}_\alpha \cup \mathcal{D}_{una} \models \mathcal{F}_\alpha \cup \mathcal{D}_{una}.$$

Similarly, by Theorem 1 again and since $\mathcal{D}_\alpha$ is a set of first-order sentences uniform in S_α it follows that $\mathcal{D} \models \mathcal{D}_\alpha \cup \mathcal{D}_{una}$. By Definition 7 it follows that

$$\mathcal{F}_\alpha \cup \mathcal{D}_{una} \models \mathcal{D}_\alpha \cup \mathcal{D}_{una}.$$

□

6. A practical case of first-order progression

In the previous section we showed that a weak progression $\mathcal{F}_\alpha$ is not correct in the general case where the progressed theory $(\mathcal{D} - \mathcal{D}_0) \cup \mathcal{F}_\alpha$ may be used to reason about unrestricted sentences about the future of S_α. In this section we will show that a weak progression $\mathcal{F}_\alpha$ is nonetheless correct wrt to a set that is wide enough to include sentences that are used in (nontrivial) practical reasoning tasks. We start by reformulating a result in [3] about the simple projection problem and then proceed with extending this result to a special case of the generalized projection problem.

6.1. Weak progression is correct wrt $\mathcal{L}^R_{S_\alpha}$

The ability to predict how the world will be after performing a sequence of actions is the basis for other more complex reasoning problems such as automated planning, scheduling, web-service composition, high-level program execution [6]. In such settings the simple projection problem refers to determining whether some condition holds in a specific point in the future, or in situation calculus terms whether the corresponding basic action theory entails a sentence that is uniform in some situation term.

Lin and Reiter [3] showed that weak progression is correct wrt sentences of the simple projection problem. The result follows by the properties of regression, an important computational mechanism that can be used to transform a sentence uniform in σ to an equivalent one about S_0. Here we introduce a slight generalization of regression and reformulate this result in our notation.

Definition 13 ($\mathcal{L}^R_\sigma$). A first-order formula ϕ in $\mathcal{L}$ is *regressable wrt the situation term* σ iff the following conditions hold:

1. every term of sort situation mentioned in ϕ is rooted at σ;
2. for every atom of the form $Poss(\alpha, \kappa)$ that appears in ϕ, α has the form $A(\vec{t})$, where A is some n-ary action function symbol;
3. ϕ does not quantify over situations;
4. ϕ does not mention the predicate symbol $\sqsubset$ and it does not mention any equality atom built on situation terms.

We define $\mathcal{L}^R_\sigma$ as the set of all first-order sentences in $\mathcal{L}$ that are regressable wrt σ.

The set of regressable formulas wrt S_0 is exactly the set of regressable formulas as defined in [5], while the set of regressable formulas wrt some ground term σ is the subset that can also be regressed down to σ.

Example 2. Consider the language of the infinite doors domain of Definition 10. The sentence $F(0, do(A, S_0)) \wedge F(0, do(B, S_0))$ is regressable wrt S_0 but it is not regressable wrt $do(A, S_0)$, while the sentence $F(0, do(A, S_0)) \wedge F(0, do(B, do(A, S_0)))$ is regressable wrt both S_0 and $do(A, S_0)$.

We introduce the regression operator $\mathcal{R}_\sigma$ for formulas that are regressable wrt σ. $\mathcal{R}_\sigma$ works exactly the same as the operator $\mathcal{R}$ defined in [5] regressing atoms according to the precondition and successor state axioms in $\mathcal{D}$, except that it only does so until a formula uniform in σ is obtained. Similar to the operator $\mathcal{R}$, the following result can be obtained for $\mathcal{R}_\sigma$.

Lemma 7. *Let $\mathcal{D}$ be a basic action theory and ϕ a first-order sentence that is regressable wrt the ground situation term σ. Then, $\mathcal{R}_\sigma(\phi)$ is a first-order sentence uniform in σ such that $\mathcal{D} \models \phi$ iff $\mathcal{D} \models \mathcal{R}_\sigma(\phi)$.*

Proof. By induction over a suitable well-founded ordering relation similar to the proof of result [14, Theorem 2]. The inductive argument and the details of the ordering relation are exactly the same except for the fact that the base case corresponds to a formula uniform in σ. □

In particular this result also holds when instead of $\mathcal{D}$ we have a progressed theory $(\mathcal{D} - \mathcal{D}_0) \cup \mathcal{F}_\alpha$.

Lemma 8. *Let $\mathcal{D}$ be a basic action theory, α a ground action term, and $\mathcal{F}_\alpha$ a weak progression of $\mathcal{D}_0$ wrt α and $\mathcal{D}$. Let ϕ be a first-order sentence that is regressable wrt S_α. Then, $\mathcal{R}_{S_\alpha}(\phi)$ is a first-order sentence uniform in S_α such that $(\mathcal{D} - \mathcal{D}_0) \cup \mathcal{D}_\alpha \models \phi$ iff $(\mathcal{D} - \mathcal{D}_0) \cup \mathcal{D}_\alpha \models \mathcal{R}_\sigma(\phi)$.*

Proof. The same proof method as the one for Lemma 7 applies. □

Using the previous result about the simple projection problem and these lemmas it is easy to show that a weak progression is correct wrt the set of first-order sentences that are regressable wrt S_α.

Lemma 9 (Weak progression is correct wrt $\mathcal{L}^R_{S_\alpha}$). *Let $\mathcal{D}$ be a basic action theory, α a ground action term, and $\mathcal{F}_\alpha$ a weak progression of $\mathcal{D}_0$ wrt α and $\mathcal{D}$. Then $\mathcal{F}_\alpha$ is a correct progression of $\mathcal{D}_0$ wrt α, $\mathcal{D}$, and $\mathcal{L}^R_{S_\alpha}$.*

Proof. Let ϕ be a first-order sentence that is regressable wrt S_α. It suffices to show that $\mathcal{D} \models \phi$ iff $(\mathcal{D} - \mathcal{D}_0) \cup \mathcal{F}_\alpha \models \phi$. By Lemma 7 it follows that the sentence $\mathcal{R}_{S_\alpha}(\phi)$ is uniform in S_α and $\mathcal{D} \models \phi$ iff $\mathcal{D} \models \mathcal{R}_{S_\alpha}(\phi)$. By the result

[3, Proposition 4.3] that a weak progression is correct wrt sentences uniform in some situation term, it follows that $\mathcal{D} \models \phi$ iff $(\mathcal{D} - \mathcal{D}_0) \cup \mathcal{F}_\alpha \models \mathcal{R}_{S_\alpha}(\phi)$. By Lemma 8 and since $\mathcal{D}$ and $(\mathcal{D} - \mathcal{D}_0) \cup \mathcal{F}_\alpha$ share the same $\mathcal{D}_{ss}$, it follows that $\mathcal{D} \models \phi$ iff $(\mathcal{D} - \mathcal{D}_0) \cup \mathcal{F}_\alpha \models \phi$. □

We now proceed to show that a weak progression is correct with respect to a much wider class of sentences that may also quantify over situations.

6.2. Weak progression is correct wrt $\mathcal{L}^Q_{S_\alpha}$

While the simple projection task refers to determining whether some condition holds in a specific point in the future, the generalized projection problem refers to conditions about several points in the future such as questions of *achievability*, i.e., "Is there a way to open all doors in the domain of infinite doors?", and *invariants*, i.e., "Will the door 0 remain closed forever after this action is performed?". We define the set $\mathcal{L}^Q_\sigma$ of first-order sentences that includes an "un-nested" form of such questions as follows.

Definition 14 ($\mathcal{L}^Q_\sigma$). Let σ be a ground situation term. Then, $\mathcal{L}^Q_\sigma$ is the smallest set such that the following conditions hold:

1. if the first-order formula $\phi(s)$ is regressable wrt s then the sentences $\phi(\sigma)$ and $\forall s(\sigma \sqsubseteq s \supset \phi(s))$ are in $\mathcal{L}^Q_\sigma$;
2. if the first-order sentences ϕ, ψ are in $\mathcal{L}^Q_\sigma$ then the sentences $\neg\phi$ and $\phi \wedge \psi$ are also in $\mathcal{L}^Q_\sigma$.

The set $\mathcal{L}^Q_\sigma$ is a subset of $\mathcal{L}^F_\sigma$ that restricts the way quantifiers for situation variables can be nested. The following example illustrates this.

Example 3. Consider the language of the infinite doors domain of Definition 10, and the sentence ϕ^* of Definition 12:

$$\exists x \forall s(S_A \sqsubseteq s \supset \neg F(x, s)),$$

ϕ^* is an example of a sentence in $\mathcal{L}^F_{S_A}$ but not in $\mathcal{L}^Q_{S_A}$. Now let ϕ be the following sentence:

$$\forall s(S_A \sqsubseteq s \supset \exists x \neg F(x, s)).$$

Observe that ϕ is in $\mathcal{L}^F_{S_A}$ and it is also in $\mathcal{L}^Q_{S_A}$. The same holds for $\neg\phi$ and boolean combinations of similar invariants that allow quantification over objects to appear only inside the scope of the quantification over situations.

The intuition is that a weak progression is correct with respect to the sentences in $\mathcal{L}^Q_\sigma$, where σ is the situation that $\mathcal{D}_0$ is progressed to. In other words, for every sentence ϕ in $\mathcal{L}^Q_{S_\alpha}$, $(\mathcal{D}-\mathcal{D}_0)\cup\mathcal{F}_\alpha \models \phi$ iff $\mathcal{D}\models\phi$. In order to prove this we first identify a sufficient condition that refers to the models of the two theories. This is stated formally in following lemma:

Lemma 10. *Let $\mathcal{D}$ be a basic action theory, α a ground action term, $\mathcal{F}_\alpha$ a weak progression of $\mathcal{D}_0$ wrt α and $\mathcal{D}$, and Δ a set of first-order sentences in $\mathcal{L}$. Let M be a model of $\mathcal{D}$ and M' a model of $(\mathcal{D}-\mathcal{D}_0)\cup\mathcal{F}_\alpha$. If the following holds for all M, M':*

> *for all first-order sentences ϕ uniform in S_α, $M\models\phi$ iff $M'\models\phi$,*
> *implies that for all ϕ in Δ, $M\models\phi$ iff $M'\models\phi$,*

then it follows that: for all $\phi\in\Delta$, $\mathcal{D}\models\phi$ iff $(\mathcal{D}-\mathcal{D}_0)\cup\mathcal{F}_\alpha\models\phi$.

This lemma is very important as it specifies a method for proving that a weak progression is correct with respect to a class of sentences Δ. Essentially, it reduces the question that the two theories entail the same set of sentences in Δ provided they entail the same set of sentences uniform in S_α, to a simpler question about the models of the theories, i.e., any two models of the theories satisfy the same set of sentences in Δ provided they satisfy the same set of sentences uniform in S_α.

The proof of Lemma 10 can be found in Appendix C along with two other results that are needed. In particular, the proof relies on the existing result that the second-order part of a basic action theory, namely the set $\mathcal{D}_{fnd}$, can be safely omitted when reasoning about formulas uniform in some situation argument, as well as a new non-trivial result (Lemma 15) that involves a non-constructive argument and the Compactness Theorem of first-order logic.

So, with Lemma 10 at hand, in order to show that a weak progression is correct with respect to $\mathcal{L}^Q_\sigma$ it suffices to prove the following lemma:

Lemma 11. *Let $\mathcal{D}$ be a basic action theory, α a ground action term, σ the situation term S_α and $\mathcal{F}_\alpha$ a weak progression of $\mathcal{D}_0$ wrt α and $\mathcal{D}$. Let M be a model of $\mathcal{D}$ and M' be a model of $(\mathcal{D}-\mathcal{D}_0)\cup\mathcal{F}_\alpha$ such that for all sentences ϕ uniform S_α, $M\models\phi$ iff $M'\models\phi$. Then, for all ϕ in $\mathcal{L}^Q_\sigma$, $M\models\phi$ iff $M'\models\phi$.*

Proof. By induction on the construction of formulas ϕ in $\mathcal{L}^Q_\sigma$. The only interesting part is the base of the induction where we have two cases: i) ϕ is regressable wrt S_α; and ii) ϕ is $\forall s(S_\alpha \sqsubseteq s \supset \psi(s))$, where $\psi(s)$ is regressable wrt s. The first case follows from Lemma 9. For the second case we will use a trick to deal with the quantification over situations and reduce it to the first case. We show the ($\Rightarrow$) direction by contradiction and the other one follows similarly.

Let $M \models \forall s(S_\alpha \sqsubseteq s \supset \psi(s))$ where $\psi(s)$ is regressable wrt s and suppose that $M' \not\models \forall s(S_\alpha \sqsubseteq s \supset \psi(s))$. It follows that there is an element q of the situation domain such that $M', \mu_q^s \models S_\alpha \sqsubseteq s \wedge \neg\psi(s)$. Since M' satisfies the foundational axioms $\mathcal{D}_{fnd}$, this element q is reachable from the denotation of S_α by a finite number of applications of the function do. In particular let $e_1, \ldots, e_n$ be elements of the action domain such that $do^{M'}(\langle e_1, \ldots, e_n\rangle, S_\alpha^{M'}) = q$. It follows that $M' \models \gamma$, where γ is the following sentence:

$$\exists a_1 \cdots \exists a_n \neg\psi(do(\langle a_1, \ldots, a_n\rangle, S_\alpha)).$$

By the hypothesis $\psi(s)$ is regressable wrt s and so γ is regressable wrt S_α. By case i) it follows that $M \models \gamma$. Since M satisfies the foundational axioms $\mathcal{D}_{fnd}$, it follows that $M \models \exists s(S_\alpha \sqsubseteq s \wedge \neg\psi(s))$ or equivalently $M \not\models \forall s(S_\alpha \sqsubseteq s \supset \psi(s))$ which is a contradiction. Thus our assumption is wrong and $M' \models \forall s(S_\alpha \sqsubseteq s \supset \psi(s))$. □

Now we are ready to state and prove the main theorem of the section.

Theorem 3 (Weak progression is correct wrt $\mathcal{L}^Q_{S_\alpha}$). *Let $\mathcal{D}$ be a basic action theory, α a ground action term, and $\mathcal{F}_\alpha$ a weak progression of $\mathcal{D}_0$ wrt α and $\mathcal{D}$. Then $\mathcal{F}_\alpha$ is a correct progression of $\mathcal{D}_0$ wrt α, $\mathcal{D}$, and $\mathcal{L}^Q_{S_\alpha}$.*

Proof. Let Δ be $\mathcal{L}^Q_{S_\alpha}$. The theorem follows by Lemma 10 and Lemma 11. □

This result shows that even though it may be the case that a first-order strong progression cannot be found, we can still use a (first-order) weak progression for our reasoning purposes and it is guaranteed that it will be correct wrt a wide class of sentences. This result establishes that this form of progression is actually more useful than what was originally believed.

7. Related Work

The notion of progression for basic action theories was first introduced by Lin and Reiter in their seminal paper [3]. The version we use here is due to Vassos *et al.* [15] that follows the same intuitions but is based on logical equivalence instead of model theoretical notions. For a more detailed discussion about the syntactic and aesthetic differences between the definitions of progression as they appear in [3], [5], and in this paper, the reader is referred to Appendix A.

There is a lot of work in specifying restrictions on the theories so that a correct progression is always definable in first-order logic. Lin and Reiter [3] were first to investigate syntactical restrictions on the successor state axioms as they introduced the *context-free* assumption for actions and showed that a first-order strong progression can be computed. Liu and Levesque [16] introduced the *local-effect* assumption for actions where they proposed a weaker version of progression that is logically incomplete, but remains practical. Vassos *et al.* [15] later showed that under this assumption a first-order strong progression can be computed by updating a finite initial knowledge base. Some more recent work shows that a first-order strong progression can be achieved under conditions also for actions that go beyond the local-effect assumption [17, 18, 19], as well as for a special form of knowledge bases that include functional fluents [20], and in an epistemic setting [21].

Other people have looked into definitions for the progression of basic action theories under different assumptions. Liu and Levesque [16] study the special case where the domain of discourse is fixed to a countable set of named objects, while Claßen and Lakemeyer [22] focus on the $\mathcal{ES}$ variant of the situation calculus. Outside of the situation calculus but in a similar logical formalism, Thielscher [23] defines a dual representation for the basic action theories based on state update axioms that explicitly define the direct effects of each action. Unlike our work where the initial knowledge base is replaced by an updated version, there, the update relies on expressing the changes using constraints which may need to be conjoined to the original knowledge base. Finally, a similar but weaker result is due to Shirazi and Amir [24]. Shirazi and Amir proposed *logical filtering* as a way to progress the initial knowledge base and proved that their method is correct for answering queries of a certain form.

Finally, with respect to the proof for Conjecture 1, it should be noted that the notion of unnamed objects was also used in a different way in [3] to show that a first-order strong progression does not always exist.

8. Concluding remarks

In this paper we focused on a technical problem in the field of reasoning about action and change in the context of the situation calculus. We examined two definitions that exist in the literature for the notion of progression, namely the definition of a strong progression that is based on a second-order construct, and that of a weak progression that is based on the first-order entailments of the original theory.

We were able to prove a major result (Theorem 2) that resolves a problem that has been open since it was first identified by Lin and Reiter in [3], namely that a weak progression is too weak, as it is not a correct progression in the general case. The consequence of this result is that in the general case we cannot always find a first-order progression that is also a correct progression: a strong progression may necessarily be second-order while a weak progression may be incorrect.

This is not a surprising result as every basic action theory includes a second-order inductive axiom that is necessary when we reason about all the possible future situations. What was not clear until now, though, is how this inductive axiom may be used implicitly to construct a first-order sentence that quantifies over situations in a way that captures a second-order property that cannot be expressed in any first-order sentence that does not quantify over situations.

Nonetheless, in practice we are not only interested in the most general case for the reasoning tasks we can perform with basic action theories. In fact, in special cases the simpler definition of weak progression may be a preferred option as long as it is guaranteed that it is correct. In this paper we were also able to prove an important positive result toward this direction.

Our result (Theorem 3) shows that a weak progression is always correct wrt reasoning about a large class of sentences, including some that quantify over situations, in particular sentences that express questions of achievability and invariants. This is an important result that shows that even though it may be the case that a first-order strong progression cannot be found, we can still use a (first-order) weak progression for our reasoning purposes and it is guaranteed that it will be correct wrt a wide class of sentences. Moreover, we provided a general method for proving the correctness of weak progression that can be used under different assumptions (Lemma 10).

References

[1] S. Vassos, H. Levesque, On the progression of situation calculus basic action theories: Resolving a 10-year-old conjecture, in: Proceedings of the Twenty-Third AAAI Conference on Artificial Intelligence, AAAI'08, AAAI Press, Chicago, Illinois, USA, 2008, pp. 1004–1009.

[2] F. Lin, R. Reiter, How to progress a database II: the STRIPS connection, in: Proceedings of the 14th International Joint conference on Artificial Intelligence - Volume 2, IJCAI'95, Morgan Kaufmann Publishers Inc., San Francisco, CA, USA, 1995, pp. 2001–2007.

[3] F. Lin, R. Reiter, How to progress a database, Artificial Intelligence 92 (1997) 131–167.

[4] J. McCarthy, P. J. Hayes, Some philosophical problems from the standpoint of artificial intelligence, Machine Intelligence 4 (1969) 463–502.

[5] R. Reiter, Knowledge in Action. Logical Foundations for Specifying and Implementing Dynamical Systems, MIT Press, 2001.

[6] R. Reiter, Proving properties of states in the situation calculus, Artificial Intelligence 64 (1993) 337–351.

[7] F. Lin, Chapter: Situation calculus, in: V. Lifschitz, F. van Harmelen, F. Porter (Eds.), Handbook of Knowledge Representation, Elsevier, 2007.

[8] H. B. Enderton, A Mathematical Introduction to Logic, Academic Press, Inc., Orlando, Florida, 1972.

[9] E. Mendelson, Introduction to Mathematical Logic, Chapman & Hall/CRC, fourth edition, 1997.

[10] R. E. Fikes, N. J. Nilsson, Strips: A new approach to the application of theorem proving to problem solving, Artificial Intelligence 2 (1971) 189–208.

[11] E. P. D. Pednault, Toward a mathematical theory of plan synthesis, Ph.D. thesis, Stanford University, Stanford, CA, USA, 1987.

[12] P. Peppas, N. Foo, M. Williams, On the expressibility of propositions, Logique et Analyse 139–149 (1995) 251–272.

[13] A. Gabaldon, Non-markovian control in the situation calculus, in: Proceedings of the Eighteenth National Conference on Artificial Intelligence, AAAI'02, AAAI Press, Menlo Park, CA, USA, 2002, pp. 519–524.

[14] F. Pirri, R. Reiter, Some contributions to the metatheory of the situation calculus, Journal of the ACM 46 (1999) 261–325.

[15] S. Vassos, G. Lakemeyer, H. Levesque, First-order strong progression for local-effect basic action theories, in: Proceedings of the Eleventh International Conference on Principles of Knowledge Representation and Reasoning, KR'08, AAAI Press, 2008, pp. 662–272.

[16] Y. Liu, H. J. Levesque, Tractable reasoning with incomplete first-order knowledge in dynamic systems with context-dependent actions, in: Proceedings of the 19th international joint conference on Artificial intelligence, IJCAI'05, Morgan Kaufmann Publishers Inc., Edinburgh, Scotland, 2005, pp. 522–527.

[17] S. Vassos, S. Sardina, H. Levesque, Progressing basic action theories with non-local effect actions, in: Proceedings of the 9th International Symposium on Formalizations of Commonsense Reasoning, pp. 135–140.

[18] Y. Liu, G. Lakemeyer, On first-order definability and computability of progression for local-effect actions and beyond, in: Proceedings of the 21st international jont conference on Artifical intelligence, IJCAI'09, Morgan Kaufmann Publishers Inc., 2009, pp. 860–866.

[19] S. Vassos, S. Sardina, A Database-type Approach for Progressing Action Theories with Bounded Effects, College Publications.

[20] V. Belle, G. Lakemeyer, On progression and query evaluation in first-order knowledge bases with function symbols, in: Proceedings of the 22nd International Joint Conference on Artificial Intelligence, IJCAI'11, IJCAI/AAAI, 2011, pp. 744–749.

[21] Y. Liu, X. Wen, On the progression of knowledge in the situation calculus, in: Proceedings of the 22nd International Joint Conference on Artificial Intelligence, IJCAI'11, pp. 976–982.

[22] J. Classen, G. Lakemeyer, A Semantics for ADL as Progression in the Situation Calculus, in: J. Dix, A. Hunter (Eds.), Proceedings of the 11th Workshop on Nonmonotonic Reasoning, NMR'06, Institut für Informatik, TU Clausthal, 2006, pp. 334–341.

[23] M. Thielscher, From situation calculus to fluent calculus: State update axioms as a solution to the inferential frame problem, Artificial Intelligence 111 (1999) 277–299.

[24] A. Shirazi, E. Amir, First-order logical filtering, Artificial Intelligence 175 (2011) 193–219.

Appendix A. A note on the various definitions of progression in the literature

Lin and Reiter were first to formally characterize when a set $\mathcal{D}_\alpha$ qualifies as a progression of $\mathcal{D}_0$ [3, Definition 4.1]. In this seminal paper Lin and Reiter gave a model-theoretic definition for $\mathcal{D}_\alpha$ and proved that their definition is always correct. In their definition they required that the models of $\mathcal{D}_\alpha$ have a specific relation with the models of $\mathcal{D}$ as far as the situation S_α is concerned.

An important detail about the definition of progression by Lin and Reiter is that it is based on a slightly different version of the basic action theories than the ones we consider nowadays. In particular, the successor state axioms in [3] had the following form:

$$Poss(a, s) \supset F(\vec{x}, do(a, s)) \equiv \Phi(\vec{x}, a, s),$$

that is, it should be the case that the action α is *executable* in S_0 in order for S_α to be affected in any way. Therefore, the model-theoretic definition of progression they provided also accounted for the occurrence of *Poss* in the successor state axioms. For example, if α is not possible in S_0 then the models of $\mathcal{D}$ are such that S_0 and S_α are identical. It follows that the models of $\mathcal{D}_\alpha$ need to reflect this as well.

Another subtle detail is that the model-theoretic definition by Lin and Reiter requires that $\mathcal{D}_\alpha$ entails $\mathcal{D}_{una}$. This requirement and the effect of *Poss* is illustrated in their result about the second-order definability of progression [3, Theorem 2]: they were able to prove that when $\mathcal{D}_0$ is a finite set, then the set $\mathcal{D}_{una}$ along with a particular second-order sentence that mentions *Poss* always qualifies as a progression of $\mathcal{D}_0$. The definition of progression of [3] also implies that any two progressions of $\mathcal{D}_0$ are logically equivalent. As a result it follows that $\mathcal{D}_\alpha$ necessarily includes $\mathcal{D}_{una}$ or a logically equivalent representation of $\mathcal{D}_{una}$.

In this paper we chose to define the progression of $\mathcal{D}_0$ using the second-order sentence that was introduced by Lin and Reiter. The reason is that it is often easier to work with a second-order sentence than a model-theoretic relation between theories. In order to use their result though, we had to remove the dependency on *Poss* as in the definition of the basic action theories that we use the predicate *Poss* is not mentioned in the successor state axioms. This was in fact easy as we only had to remove the left hand side of the implication of the second-order sentence of [3]. Intuitively this implies that the situations progress as if $Poss(a, s)$ is always true.

Moreover, we find that it is not aesthetically nice to include $\mathcal{D}_{una}$ in $\mathcal{D}_\alpha$ knowing that $\mathcal{D}_{una}$ is already included in $\mathcal{D}$. In other words, since $\mathcal{D}_\alpha$ includes a logically equivalent representation of $\mathcal{D}_{una}$, the updated basic action theory $(\mathcal{D}-\mathcal{D}_0)\cup\mathcal{D}_\alpha$ essentially includes $\mathcal{D}_{una}$ twice. In order to prevent this we also removed the dependency on $\mathcal{D}_{una}$ so that the progression of $\mathcal{D}_0$ corresponds to a more intuitive set that does not include information that is implied by $\mathcal{D}_{una}$ unless it is necessary for specifying the situation S_α.

So, the definition of a strong progression is essentially the original definition of progression by Lin and Reiter [3] applied to the newer form of basic action theories that do not use *Poss* in the successor state axioms. In fact, the correctness of strong progression follows from the results in [3] in the sense that we can follow the same reasoning as in the proofs of Lin and Reiter with only slight changes that correspond to the two points we mentioned about *Poss* and $\mathcal{D}_{una}$.

There is one more definition in the literature that is also very related to the original definition by Lin and Reiter, namely the definition of progression that appears in [5, Definition 9.1.1]. This is also model-theoretic, more compact than the original definition by Lin and Reiter, and also accounts for the aesthetic issue we noted about the set $\mathcal{D}_{una}$. This definition of $\mathcal{D}_\alpha$ requires that the models of $(\mathcal{D}-\mathcal{D}_0)\cup\mathcal{D}_\alpha$ instead of the models of $\mathcal{D}_\alpha$ have a specific relation with the models of $\mathcal{D}$ as far as the situations in the future of S_α are concerned. Nonetheless the definition in [5] suffers from the following problem: it allows two sets that are not logically equivalent to both qualify as a progression of $\mathcal{D}_0$. Unfortunately we can no longer then assume that $\mathcal{D}_\alpha$ is unique up to logical equivalence which in some sense means that $\mathcal{D}_\alpha$ is not well-defined.

For instance consider the case where $\mathcal{D}_0$ is the empty set and consider the action A that makes $F(s)$ true and does not affect any other fluent. The successor state axiom for F is then the following:

$$F(do(a,s)) \equiv (a = A \vee F(s)).$$

According to the definition in [5] it is not too difficult to show that the set

$$\{F(do(A,S_0))\}$$

is a progression of $\mathcal{D}_0$, which is indeed the intended progression of $\mathcal{D}_0$. Unfortunately, there is one more set that qualifies as a progression of $\mathcal{D}_0$, one that is not intended to qualify as a progression. Let $\mathcal{D}_\alpha$ be the empty set. Then it is trivial to show that $\mathcal{D}_\alpha$ is a progression of $\mathcal{D}_0$ according to the definition in [5] simply because $\mathcal{D}$ and $(\mathcal{D} - \mathcal{D}_0) \cup \mathcal{D}_\alpha$ are identical as both $\mathcal{D}_0$ and $\mathcal{D}_\alpha$ are empty. So, in this simple example the empty set and the set $\{F(do(A, S_0))\}$ both qualify as a progression of $\mathcal{D}_0$ with respect to the definition in [5] but they are clearly not logically equivalent. Note that this cannot arise in the case of strong progression or the original definition by Lin and Reiter.

Appendix B. Proof of Lemma 4

We first present a lemma that shows a particular relation between the models of two first-order theories Σ_1, Σ_2 and their entailments with respect to a set of first-order sentences Γ. The intuition is that this lemma also apply when Σ_1 is a basic action theory $\mathcal{D}$ without the set $\mathcal{D}_{fnd}$ of the foundational axioms and Σ_2 is a progression of $\mathcal{D}_0$ according to some appropriate definition.

Lemma 12. *Let $\Sigma_1, \Sigma_2, \Gamma$ be sets of first-order or second-order sentences of the situation calculus language $\mathcal{L}$. If for every model M_1 of Σ_1, there is a model M_2 of Σ_2 such that M_1 and M_2 satisfy the same set of sentences in Γ, then the following holds: for all ϕ in Γ, if $\Sigma_2 \models \phi$ then $\Sigma_1 \models \phi$.*

Proof. Let ϕ be an arbitrary sentence in Γ and assume that $\Sigma_2 \models \phi$. Let M_1 be an arbitrary model of Σ_1. By the hypothesis of the lemma it follows that there is a model M_2 of Σ_2 such that M_1 and M_2 satisfy the same set of sentences in Γ. By the assumption that $\Sigma_2 \models \phi$ it follows that $M_2 \models \phi$, and since ϕ is in Γ it follows that $M_1 \models \phi$. Since M_1 was arbitrary it follows that $\Sigma_1 \models \phi$. □

Lemma 4 *Let $\mathcal{D}$ be the basic action theory of the infinite doors domain of Definition 10 and $\mathcal{F}_\alpha$ the following set of first-order sentences:*

$$\{\forall x(x = 0 \equiv F(x, S_A)),\ (5),\ (6),\ (7)\}.$$

Then, $\mathcal{F}_\alpha$ is a weak progression of $\mathcal{D}_0$ wrt the ground action A and $\mathcal{D}$.

Proof. By Definition 7 we need to show that for all first-order sentences ϕ uniform in S_A, $\mathcal{F}_\alpha \cup \mathcal{D}_{una} \models \phi$ iff $\mathcal{D} \models \phi$.

For the ($\Rightarrow$) direction we proceed as follows. The theory $\mathcal{D}$ entails the successor state axiom for F therefore also entails the sentence we get by replacing s by S_0 and a by A:

$$\mathcal{D} \models \forall x\ F(x, do(A, S_0)) \equiv A = A \wedge x = 0 \vee \\ A = B \wedge \neg F(x, S_0) \wedge \exists y(x = n(y) \wedge F(y, S_0)),$$

By the axioms for equality and the sentence (4), i.e., $A \neq B$, it follows that

$$\mathcal{D} \models \forall x\ x = 0 \equiv F(x, S_A).$$

It is clear that $\mathcal{D} \models \{(5), (6), (7)\}$ as these are axioms of $\mathcal{D}$. Therefore

$$\mathcal{D} \models \{\forall x(x = 0 \equiv F(x, S_A)),\ (5), (6), (7)\}.$$

It follows that $\mathcal{D} \models \mathcal{F}_\alpha \cup \mathcal{D}_{una}$. Now let ϕ be a first-order sentence uniform in S_A and assume that $\mathcal{F}_\alpha \cup \mathcal{D}_{una} \models \phi$. Then since $\mathcal{D} \models \mathcal{F}_\alpha \cup \mathcal{D}_{una}$ and $\mathcal{F}_\alpha \cup \mathcal{D}_{una} \models \phi$ it follows that $\mathcal{D} \models \phi$.

For the ($\Leftarrow$) direction the case is more complicated. Let Σ_1 be the set $\mathcal{F}_\alpha \cup \mathcal{D}_{una}$, Σ_2 the set $\mathcal{D}$, and Γ the set of first-order sentences uniform in S_A. Then by Lemma 12 it suffices to show that for every model M of $\mathcal{F}_\alpha \cup \mathcal{D}_{una}$, there is a model M' of $\mathcal{D}$ such that M and M' satisfy the same set of sentences in Γ. Let M be an arbitrary model of $\mathcal{F}_\alpha \cup \mathcal{D}_{una}$. Based on Definition 11 of named and unnamed objects we distinguish two cases for: i) M has at least one element in the object domain that is unnamed, and ii) all the elements of the object domain of M are named. For both cases we will show that there is a model M' of $\mathcal{D}$ such that M and M' satisfy the same set of sentences in Γ. For the first case we appeal to a natural construction, while the second case will be reduced to the first one by appropriate application of the Löwenheim-Skolem theorem.

Case i): $M \models \mathcal{F}_\alpha \cup \mathcal{D}_{una}$ and M has at least one element in the object domain that is unnamed. We construct the model M' as follows.

1. M' has the same domains for sort object and action as M.
2. M' has a domain of strings Sit for the situation sort that is defined as follows. Sit includes the string "S_0" and it is the smallest set that is closed under the string operation that produces the string "$do(a, s)$", where a can be either the string "A" or the string "B" and s is any string in Sit. M' interprets $\sqsubset$ as the sub-string relation between strings.
3. M' interprets $0, n, A, B$ exactly as M.
4. M' interprets S_0, do such that every situation term σ in $\mathcal{L}$ is interpreted to the corresponding string "σ" in Sit.
5. $M', \mu^x_q \models F(x, S_0)$, for all q in the object domain of M' such that q is named.

6. $M', \mu_q^x \not\models F(x, S_0)$, for all q in the object domain of M' such that q is unnamed.
7. For all q in the object domain of M' (or equivalently the object domain of M), $M', \mu_q^x \models F(x, S_A)$ iff $M, \mu_q^x \models F(x, S_A)$.
8. For all the elements of Sit other than "S_0" and "$do(A, S_0)$" let the interpretation of $F(x, s)$ be the one that is defined by the successor state axiom for F and the interpretation of $F(x, S_0)$, $F(x, S_A)$.
9. $M' \models \forall a \forall s Poss(a, s)$.

By the construction of M' and in particular by the points 1,3,7 and by induction on the construction of the first-order formulas uniform in S_A it follows that for all sentences ϕ in Γ, $M \models \phi$ iff $M' \models \phi$. We now proceed to show also that M' is a model of $\mathcal{D}$, that is, M' satisfies the sentences (1) to (7) as they appear in Definition 10, as well as the foundational axioms $\mathcal{D}_{fnd}$ for basic action theories.

- Sentences (1), (2): $Poss(A, s) \equiv true$, $Poss(B, s) \equiv true$.

  By point 9 it follows that $M' \models (1)$ and $M' \models (2)$.

- Sentence (3):

  $$F(x, do(a, s)) \equiv a = A \wedge x = 0 \vee \\ a = B \wedge \neg F(x, s) \wedge \exists y (x = n(y) \wedge F(y, s)).$$

  Let $\Phi(x, a, s)$ be the formula on the right hand side of the logical symbol $\equiv$ in the sentence (3). By the point 8 it follows that for all variable assignments μ except for μ' which interprets $do(a, s)$ as "$do(A, S_0)$", $M', \mu \models \forall x. F(x, do(a, s)) \equiv \Phi(x, a, s)$. By the point 7 and since M is a model of $\mathcal{F}_\alpha \cup \mathcal{D}_{una}$ it follows that $M' \models \forall x. x = 0 \equiv F(x, do(A, S_0))$ which implies that $M', \mu' \models \forall x. F(x, do(a, s)) \equiv \Phi(x, a, s)$. Therefore, $M' \models (3)$.

- Sentences (4), (5), (6), (7): $A \neq B$, $\forall a\ a = A \vee a = B$, $\forall x\ x \neq 0 \equiv \exists y\ n(y) = x$, $\forall x \forall y\ n(x) = n(y) \supset x = y$.

  $M \models \mathcal{F}_\alpha \cup \mathcal{D}_{una}$ therefore $M \models \{(4), (5), (6), (7)\}$. By the points 1,3 it follows that $M' \models \{(4), (5), (6), (7)\}$.

- Sentence (8): $F(0, S_0) \wedge \forall x\, (F(x, S_0) \supset F(n(x), S_0))$.

  By the points 5 and 6 it follows that $M' \models (8)$.

- Sentence (9): $\exists x\ \neg F(x, S_0)$.

  By the point 6 and the fact that in the case i) that we consider here there is an unnamed object in the domain, it follows that $M' \models (9)$.

- $\mathcal{D}_{fnd}$. By the point 2 it follows that $M' \models \mathcal{D}_{fnd}$.

Thus, $M' \models \mathcal{D}$.

Case ii): $M \models \mathcal{F}_\alpha \cup \mathcal{D}_{una}$ and all the elements of the object domain of M are named. For this case we will use the Löwenheim-Skolem theorem of first-order logic to show that there is structure M_c that satisfies the same set of sentences in Γ as M but also has unnamed objects. We can then construct M' in the same way as for the *Case i)* so that M' is a model of $\mathcal{D}$ and satisfies the same set of sentences in Γ as M_c (and thus satisfies the same set of sentences in Γ as M). Even though the intuition for the construction of M_c is relatively simple, the actual construction involves several tedious intermediate steps. In particular, in order to use the (upward) Löwenheim-Skolem theorem we'll need to transform the three-sorted model M into a normal one-sorted model M_1 in a way that the satisfaction of formulas is preserved, then apply the theorem to get an elementarily equivalent model M_2 with unnamed objects, and then transform M_2 to a model M_c of the three-sorted situation calculus language. The formal details follow.

Let $\mathcal{L}^*$ be the (one-sorted) first-order language that consists of the constant symbols $0, A, B$, the unary function symbol n, the unary predicate symbols P, Q, and the countably infinite number of variables $a, a_1, a_2, \ldots$ and $x, x_1, x_2, \ldots$. The predicate symbol P will be used to encode the truth of the fluent atoms of the form $F(t, S_A)$ and the predicate Q will be used to specify a subset of the domain that will be used as the action domain in $\mathcal{L}^*$. The variables $x, x_1, \ldots$ will be used to quantify over the whole domain while the variables $a, a_1, \ldots$ will be used to quantify over the set of objects that lie in the extension of Q. In particular let $\mathcal{L}^*_{S_A}$ be the smallest set of formulas in $\mathcal{L}^*$ such that the following conditions hold:

- for every predicate atom $P(t)$ such that the term t is of the form $n^k(0)$ or t is one of the variables $x, x_1, x_2, \ldots$, $P(t)$ is in $\mathcal{L}^*_{S_A}$;

- for every equality atom $t_1 = t_2$ such that t_1, t_2 are terms of the form $n^k(0)$ or one of the variables $x, x_1, x_2, \ldots$, $t_1 = t_2$ is in $\mathcal{L}^*_{S_A}$;

- for every equality atom $t_1 = t_2$ such that t_1, t_2 are one of the constants A, B or one of the variables $a, a_1, a_2, \ldots$, $t_1 = t_2$ is in $\mathcal{L}^*_{S_A}$;
- if ϕ, ψ are in $\mathcal{L}^*_{S_A}$ then $\neg\phi$, $\phi \wedge \psi$, $\forall x\phi$, and $\forall a\ Q(a) \supset \phi$ are in $\mathcal{L}^*_{S_A}$.

For every formula ϕ in Γ let ϕ^* be ϕ with each fluent atom of the form $F(t, S_A)$ replaced by the atom $P(t)$ and each action quantifier of the form $\forall a\psi$ replaced by $\forall a Q(a) \supset \phi$. It is not difficult to show that if ϕ is in Γ then ϕ^* is in $\mathcal{L}^*_{S_A}$ and similarly that we can do the inverse transformation in order to obtain a formula in Γ from a formula in $\mathcal{L}^*_{S_A}$.

We now construct a structure M_1 of the language $\mathcal{L}^*$ such that the following hold.

1. M_1 has the object domain of M as its domain and interprets $n, 0$ in the same way as M.
2. M_1 interprets A as the denotation of 0, B to the denotation of $n(0)$, and the extension of Q is such that $Q(a)$ is true only for the denotations of 0 and $n(0)$.
3. For all q in the domain of M_1 (or equivalently the object domain of M), $M_1, \mu^x_q \models P(x)$ iff $M, \mu^x_q \models F(x, S_A)$.

Let g_1 be a function from the action domain of M to the set $\{A^{M_1}, B^{M_1}\}$ such that g_1 maps the denotation of A in M to the denotation of A in M_1 and similarly for B. Since M models the sentences (4) and (5) it follows that the domain of actions for M has exactly two elements and so g_1 is a bijection. Let h_1 be the extension of g_1 that also maps each of the elements of the object domain of M to the identical element in the domain of M_1. The mapping h_1 is a bijection with the obvious definition for the inverse mapping. By induction then on the construction of the first-order formulas uniform in S_A it follows that for all first-order formulas ϕ uniform in S_A, $M, \mu \models \phi$ iff $M_1, h_1(\mu) \models \phi^*$. Therefore it follows that for all sentences ϕ in Γ,

$$M \models \phi \text{ iff } M_1 \models \phi^*. \tag{B.1}$$

By the (upward) Löwenheim-Skolem theorem of first-order logic it follows that there is a structure M_2 of the language $\mathcal{L}^*$ such that M_1 and M_2 are elementarily equivalent (i.e. they satisfy the same set of sentences in $\mathcal{L}^*$) but M_1 and M_2 are not isomorphic; in particular the domain of M_2 has a greater cardinality than the domain of M_1. So there is a model M_2 of the language $\mathcal{L}^*$ such that the following conditions hold.

1. The cardinality of the domain of M_2 is greater than the cardinality of the domain of M_1.
2. For all $\phi^* \in \mathcal{L}^*$,
$$M_1 \models \phi^* \text{ iff } M_2 \models \phi^*. \tag{B.2}$$

By the assumption we did for the *Case ii)* all the objects in the domain of M are named by some term in $\mathcal{L}$ of the form $n^k(0)$ and so by the point 1 in the construction of M_1 and the fact that $\mathcal{L}^*$ includes the symbols $n, 0$ it follows that all the objects in the domain of M_1 are also named by some term in $\mathcal{L}^*$. Since the set of all terms of $\mathcal{L}^*$ is countable this implies that the domain of M_1 is also countable. Now since the cardinality of the domain of M_2 is greater than the cardinality of domain of M_1 this implies that the domain of M_2 is uncountable. Since the set of all terms of $\mathcal{L}^*$ is countable it then follows that there is at least one unnamed object in M_2.

Based on the $\mathcal{L}^*$-structure M_2 we will now construct an $\mathcal{L}$-structure M_c that satisfies the same set of sentences in Γ as M but also has at least one unnamed object. We construct M_c as follows.

1. M_c has the domain of M_2 as the object domain and interprets $n, 0$ exactly as M_2.
2. M_c has the domain $\{A^{M_2}, B^{M_2}\}$ as the action domain and interprets A, B exactly as M_2.
3. M_c has the set of strings {"S"} as the domain for sort situation and interprets all situation terms as the string "S".
4. For all q in the object domain of M_c (or equivalently the domain of M_2), $M_c, \mu_q^x \models F(x, S_A)$ iff $M_2, \mu_q^x \models P(x)$.

The set of ground terms of sort object in $\mathcal{L}$ is a subset of the set of ground terms of $\mathcal{L}^*$. By the fact that there is at least one unnamed object in the domain of M_2 and point 1 in the construction of M_c it follows that there is at least one element of the object domain of M_c that is unnamed. So in order to complete the proof we only need to show that M_c satisfies the same set of sentences in $\mathcal{L}^*_{S_A}$ as M. Let g_2 be a function from the action domain of M_c to the set $\{A^{M_2}, B^{M_2}\}$ such that g_2 maps the denotation of A in M_c to the denotation of A in M_2 and the same for B. Similarly to the definition of h_1 let h_2 be the extension of g_2 that also maps each of the elements of the object domain of M_c to the identical element in the domain of M_2. The mapping h_2 is a bijection with the obvious definition for the inverse mapping. By induction then on the construction of first-order formulas uniform in S_A it follows that for all first-order formulas uniform in S_A, $M_c, \mu \models \phi$ iff $M_2, h_2(\mu) \models \phi^*$. Therefore it follows that for every sentence ϕ in Γ,

$$M_c \models \phi \text{ iff } M_2 \models \phi^*. \tag{B.3}$$

By (B.1), (B.2), and (B.3) it then follows that for all sentences ϕ in Γ, $M \models \phi$ iff $M_c \models \phi$ which reduces the *Case ii)* to the *Case i)*. □

Appendix C. Proof of Lemma 10

Before we proceed to the proof of Lemma 10 we need to establish some results about important properties of basic action theories.

As it has been shown in [3, 14] the foundational axioms are only needed when a formula quantifies over the situation space, otherwise they can be omitted. This is an important property as the set $\mathcal{D}_{fnd}$ of the foundational axioms is the only place where a second-order axiom is used, therefore $\mathcal{D} - \mathcal{D}_{fnd}$ is a purely first-order theory. We extend this result slightly for our purposes as follows.

Lemma 13. *Let $\mathcal{D}$ be a basic action theory. Given any model M^- of $\mathcal{D} - \mathcal{D}_{fnd}$, there is a model of $\mathcal{D}$ such that the following hold:*

1. *M^- and M have the same domains for sorts action and object, and interpret all situation independent predicates and functions the same;*
2. *for any ground situation term σ, any first-order formula ϕ uniform in σ, and any variable assignment μ, $M, \mu \models \phi$ iff $M^-, \mu \models \phi$.*

Proof. By [3, Proposition 3.1] and induction on the construction of first-order formulas ϕ that are uniform in some situation term σ. □

This implies that when ϕ is uniform in some situation term σ, the set $\mathcal{D}_{fnd}$ is not needed to decide whether ϕ is entailed by the basic action theory.

Lemma 14. *Let $\mathcal{D}$ be a basic action theory. Then, for any ground situation term σ and any first-order formula ϕ uniform in σ, $\mathcal{D} \models \phi$ iff $(\mathcal{D} - \mathcal{D}_{fnd}) \models \phi$.*

The following lemma shows that if Σ_1 and Σ_2 entail the same set of sentences in Γ then for every model of one theory we can always find a model of the other such that the two models satisfy the same set of sentences in Γ. Intuitively this might seem like an obvious fact but the proof is actually not straightforward and involves a non-constructive argument that makes use of the Compactness Theorem of first-order logic.

Lemma 15. *Let $\Sigma_1, \Sigma_2, \Gamma$ be sets of first-order sentences of the situation calculus language $\mathcal{L}$ such that the following two conditions hold:*

1. *Γ is closed under logical conjunction and negation;*
2. *for all ϕ in Γ, $\Sigma_1 \models \phi$ iff $\Sigma_2 \models \phi$.*

Then for every model M_1 of Σ_1, there is a model M_2 of Σ_2 such that for all ϕ in Γ, $M_1 \models \phi$ iff $M_2 \models \phi$.

Proof. We prove by contradiction as follows. Suppose otherwise. Then there is a model M_1 of Σ_1 such that there is no model M_2 of Σ_2 so that for all sentences ϕ in Γ, $M_1 \models \phi$ iff $M_2 \models \phi$. Equivalently, there is a model M_1 of Σ_1 such that for every model M_2 of Σ_2 there is some sentence ψ in Γ so that $M_1 \models \psi$ but $M_2 \not\models \psi$.[7] Let Δ be the following set:

$$\{\psi : \psi \in \Gamma \text{ and there is a model } M_2 \text{ of } \Sigma_2 \text{ such that } M_1 \models \psi \text{ but } M_2 \not\models \psi\}.$$

Clearly Δ is consistent as $M_1 \models \Delta$. Moreover there is no model M_2 of Σ_2 that satisfies Δ since for each model M_2 there is at least one sentence ψ in Δ such that $M' \not\models \psi$. Therefore $\Sigma_2 \cup \Delta \models \Box$. By the Compactness Theorem for first-order logic we get that there is a finite subset of Δ, Δ', such that $\Sigma_2 \cup \Delta' \models \Box$. Let γ be the conjunction of all sentences in Δ'. Then $\Sigma_2 \cup \{\gamma\} \models \Box$ which implies that $\Sigma_2 \models \neg\gamma$.

Since Γ is closed under logical conjunction and negation, $\gamma \in \Gamma$ as it is a conjunction of sentences in Γ and also $\neg\gamma \in \Gamma$. By point 2 in the hypothesis Σ_1 and Σ_2 entail the same set of sentences in Γ therefore we also get that $\Sigma_1 \models \neg\gamma$. Note that this yields a contradiction because M_1 is a model of Σ_1 and M_1 satisfies γ. □

Now we can proceed to the proof of Lemma 10.

Lemma 10 *Let $\mathcal{D}$ be a basic action theory, α a ground action term, $\mathcal{F}_\alpha$ a weak progression of $\mathcal{D}_0$ wrt α and $\mathcal{D}$, and Δ a set of first-order sentences in $\mathcal{L}$. Let M be a model of $\mathcal{D}$ and M' a model of $(\mathcal{D} - \mathcal{D}_0) \cup \mathcal{F}_\alpha$. If the following holds for all M, M':*

> *for all first-order sentences ϕ uniform in S_α, $M \models \phi$ iff $M' \models \phi$,*
> *implies that for all ϕ in Δ, $M \models \phi$ iff $M' \models \phi$,*

[7] Note that if instead of this there is a sentence ψ' in Γ so that $M_1 \not\models \psi'$ but $M_2 \models \psi'$ we can just take ψ to be $\neg\psi'$.

then it follows that: for all $\phi \in \Delta$, $\mathcal{D} \models \phi$ iff $(\mathcal{D} - \mathcal{D}_0) \cup \mathcal{F}_\alpha \models \phi$.

Proof. For the ($\Leftarrow$) direction we proceed as follows. Since $\mathcal{F}_\alpha$ is a set of first-order sentences uniform in S_α and $(\mathcal{D} - \mathcal{D}_0) \cup \mathcal{F}_\alpha \models \mathcal{F}_\alpha$, by the result [3, Proposition 4.3] that a weak progression is correct wrt sentences uniform in some situation term, it follows that $\mathcal{D} \models \mathcal{F}_\alpha$. Therefore $\mathcal{D} \models (\mathcal{D} - \mathcal{D}_0) \cup \mathcal{F}_\alpha$ and as a result for all ϕ in Δ, if $(\mathcal{D} - \mathcal{D}_0) \cup \mathcal{F}_\alpha \models \phi$, then $\mathcal{D} \models \phi$.

For the ($\Rightarrow$) direction we proceed as follows. Let Σ_1 be the theory $(\mathcal{D} - \mathcal{D}_0 - \mathcal{D}_{fnd}) \cup \mathcal{F}_\alpha$, Σ_2 the theory $\mathcal{D} - \mathcal{D}_{fnd}$, and Γ the set of all the first-order sentences uniform in S_α. By the result [3, Proposition 4.3] that a weak progression is correct wrt sentences uniform in some situation term, it follows that for all ϕ in Γ, $(\mathcal{D} - \mathcal{D}_0) \cup \mathcal{F}_\alpha \models \phi$ iff $\mathcal{D} \models \phi$ and by Lemma 14 it follows that for all ϕ in Γ, $\Sigma_1 \models \phi$ iff $\Sigma_2 \models \phi$. The set Γ is clearly closed under conjunction and negation, and so by Lemma 15 it follows that for every model M_1 of Σ_1, there is a model M_2 of Σ_2 such that for all ϕ in Γ, $M_1 \models \phi$ iff $M_2 \models \phi$. Since a model of $(\mathcal{D} - \mathcal{D}_0) \cup \mathcal{F}_\alpha$ must satisfy $(\mathcal{D} - \mathcal{D}_0 - \mathcal{D}_{fnd}) \cup \mathcal{F}_\alpha$ it follows that for every model M_1 of $(\mathcal{D} - \mathcal{D}_0) \cup \mathcal{F}_\alpha$, there is a model M_2 of Σ_2 such that for all ϕ in Γ, $M_1 \models \phi$ iff $M_2 \models \phi$. By Lemma 13 it follows that for every model M_1 of $(\mathcal{D} - \mathcal{D}_0) \cup \mathcal{F}_\alpha$, there is a model M_2 of $\mathcal{D}$ such that for all ϕ in Γ, $M_1 \models \phi$ iff $M_2 \models \phi$. By the assumption of the lemma about the the models of the theories it follows that for every model M_1 of $(\mathcal{D} - \mathcal{D}_0) \cup \mathcal{F}_\alpha$, there is a model M_2 of Σ_2 such that for all ϕ in Δ, $M_1 \models \phi$ iff $M_2 \models \phi$. Now, let Σ_1' be $(\mathcal{D} - \mathcal{D}_0) \cup \mathcal{F}_\alpha$, Σ_2' to be $\mathcal{D}$. Then by Lemma 12 it follows that for all $\phi \in \Delta$, if $\mathcal{D} \models \phi$ then $(\mathcal{D} - \mathcal{D}_0) \cup \mathcal{F}_\alpha \models \phi$. □